Skew lines

Yue Kwok Choy

Problem Find the shortest distance between the lines:

$$L_1: x = 2 + 2t,\ \ y = 4 - 2t, z = 4t$$
$$L_2: x = 4 - 3t,\ \ y = 4 + 4t, z = 1 - 2t$$

1. What are skew lines?

Skew lines are two (or more) lines which non-parallel but cannot meet. Imagine you are sitting in a room facing a wall. The top horizontal ceiling line in front of you and the bottom floor line on your right hand side is an example of two skew lines. The vertical line between the front wall and the side wall on your right is the shortest distance between these skew lines.

2. Note that : $A = (2,4,0)$ is a point on L_1 and $\mathbf{r_1} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ is the direction vector of L_1.
$B = (4,4,1)$ is a point on L_2 and $\mathbf{r_2} = -3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ is the direction vector of L_2.

(a) Are L_1 and L_2 parallel?
They are not parallel, since $\mathbf{r_1}$ and $\mathbf{r_2}$ are not parallel.

(b) Can L_1 and L_2 intersect?
Equate the x values, y values and z values in L_1 and L_2.

$2 + 2t = 4 - 3t,\ \ 4 - 2t = 4 + 4t,\ \ 4t = 1 - 2t$

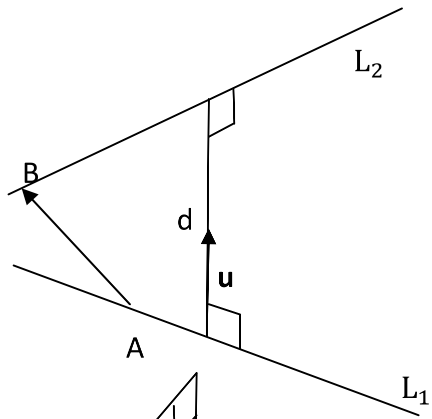
$t = \frac{2}{5}$, $t = 0$, $t = \frac{1}{6}$

Therefore these sets of equations are inconsistent and there is no intersection point.

3. $\mathbf{r_1} \times \mathbf{r_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 4 \\ -3 & 4 & -2 \end{vmatrix} = -12\mathbf{i} - 8\,\mathbf{j} + 2\,\mathbf{k}$ gives you a line perpendicular to both L_1 and L_2.

$|\mathbf{r_1} \times \mathbf{r_2}| = \sqrt{(-12)^2 + (-8)^2 + 2^2} = 2\sqrt{53}$

$\mathbf{u} = \frac{\mathbf{r_1} \times \mathbf{r_2}}{|\mathbf{r_1} \times \mathbf{r_2}|} = \frac{-12\mathbf{i} - 8\,\mathbf{j} + 2\,\mathbf{k}}{2\sqrt{53}} = \frac{-6\mathbf{i} - 4\mathbf{j} + \mathbf{k}}{\sqrt{53}}$ is a unit vector in the direction of the shortest distance.



$\mathbf{AB} = (4-2)\mathbf{i} + (4-4)\mathbf{j} + (1-0)\mathbf{k} = 2\mathbf{i} + 0\,\mathbf{j} + \mathbf{k}$

You can then move $\mathbf{AB}$ parallelly to form a triangle as shown in the bottom diagram.

Let d be the shortest distance and $\theta = \angle(\mathbf{u}, \mathbf{AB})$

$|\mathbf{u} \cdot \mathbf{AB}| = |\mathbf{u}||\mathbf{AB}|\cos\theta = |\mathbf{AB}|\cos\theta = d$

$d = |\mathbf{u} \cdot \mathbf{AB}| = \left|\frac{-6\mathbf{i} - 4\mathbf{j} + \mathbf{k}}{\sqrt{53}} \cdot (2\mathbf{i} + 0\,\mathbf{j} + \mathbf{k})\right|$

$= \left|\frac{1}{\sqrt{53}}[(-6)(2) + (-4)(0) + (1)(1)]\right| = \frac{11\sqrt{53}}{53}$

$\approx \mathbf{1.510966}$